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The two-dimensional wave pattern produced by a disturbance moving in an arbitrary direction in a density stratified liquid

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The two-dimensional internal wave pattern produced by a small disturbance moving with constant velocity along a line of arbitrary inclination in an inviscid, density stratified liquid is studied. It is shown how the far field wave pattern evolves as the inclination changes from the extreme of horizontal motion to that of vertical motion. It is found that, in general, waves propagate ahead of the disturbance.

1. Introduction

It is known that if a cylinder with horizontal axis is moved horizontally in a direction perpendicular to its generators with a velocity which is sufficiently small in a fluid of non-uniform density then a slab of fluid is pushed ahead of and dragged behind the cylinder. A schlieren photograph of this phenomenon was published by Mowbray & Rarity (1967*a*, plate 1, figure 1). It may be considered to be the internal wave pattern produced by the horizontal motion of the cylinder. If the cylinder moves in a vertical direction, there is a 'flared-skirt' wave pattern trailing behind the body; the analogous situation with axial symmetry about a vertical line, that is the wave pattern produced by a sphere moving vertically in a density stratified fluid, was investigated by Mowbray & Rarity (1967*b*), in which there appears several schlieren photographs of the wave pattern; one of these photographs appears also in Lighthill (1967). Where the direction of motion is vertical, there is no essential difference between the wave patterns produced by a sphere and a horizontal cylinder; this is not so for any other direction of motion.

The present paper analyses the two-dimensional wave pattern produced by a disturbance which moves with a constant velocity V at an angle α to the horizontal in a stably stratified fluid. It is shown how the pattern changes from the 'flared skirt' to the 'upstream wake' as α changes from $\frac{1}{2}\pi$ to zero. It is found that the far field upstream wave pattern disappears if the parameter $V\omega_0/g$ exceeds the value $2\cos\alpha$; ω_0 is the Väisälä–Brunt frequency $(-g\rho^{-1}d\rho/dy)^{\frac{1}{2}}$. The effect of the finite extent of the disturbance is also investigated. The density of the fluid is assumed to decrease exponentially with height so that the medium is homogeneous with respect to internal waves; if the medium is slightly inhomogeneous,

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the resulting wave pattern may be obtained as a deviation from the pattern in a homogeneous medium by allowing this pattern to adjust to local conditions, keeping the frequency constant.

2. Wave patterns from a moving body

If ρ is the density, p the pressure, $\mathbf{g}$ the acceleration due to gravity and $\mathbf{q}$ the velocity vector, then, in a co-ordinate system with x horizontal, y vertical and increasing upwards, fixed with respect to the body which moves with a velocity V at an angle α to the horizontal, the equations of motion are

$$\frac{D\rho}{Dt} = 0, \quad \nabla . \mathbf{q} = 0, \quad \frac{D\mathbf{q}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g},$$

where D/Dt denotes

$$\frac{\partial}{\partial t} + (u - V\cos\alpha)\frac{\partial}{\partial x} + (v - V\sin\alpha)\frac{\partial}{\partial y},$$

where we have used the condition that the density is constant along a particle path. If we introduce a stream-function ψ such that if u and v are the perturbation velocities conjugate to x and y respectively then $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$ and if we linearize about a state of rest in this co-ordinate system, then the equation for ψ has solutions proportional to $\exp(\omega_0^2 y/2g)\exp\{i(k_1x + k_2y - \omega t)\}$ provided

$$(\omega + k_1V\cos\alpha + k_2V\sin\alpha)^2 = \omega_0^2k_1^2\{k_1^2 + k_2^2 + (\omega_0^2/2g)^2\}^{-1},$$

where ω_0^2 denotes the square of the Väisälä–Brunt frequency $-g\rho^{-1}d\rho/dy$. The Boussinesq approximation has not been made; the approximation would set $\omega_0^2/2g$ identically zero in the dispersion relation above. The disturbance stream-function ψ may be represented as a double Fourier integral

$$\psi = \exp(\omega_0^2 y/2g)\iint f(k_1, k_2)\exp\{i(k_1x + k_2y)\}dk_1dk_2,$$

where f is related to the precise nature of the disturbance. The condition that the pattern be steady with respect to the body, that is $\omega \equiv 0$, yields the condition

$$K_2 = (K_2\cos\alpha + K_1\sin\alpha)(K_1^2 + K_2^2 + \beta^2)^{-\frac{1}{2}},$$

where

$$\omega_0K_1 = k_1V\sin\alpha - k_2V\cos\alpha, \quad \omega_0K_2 = k_1V\cos\alpha + k_2V\sin\alpha \quad \text{and} \quad \beta^2 = \tfrac{1}{4}V^2\omega_0^2/g^2.$$

The new wave-number component K_2 is a scaled wave-number component conjugate to the direction of motion of the body; K_1 is the wave-number component in a perpendicular direction so that (K_1, K_2) form a right-handed set. When $\alpha = 90°$, so that the body is moving vertically upwards, K_2 is conjugate to the positive y direction and K_1 to the positive x direction. We may define new co-ordinates X, Y by the relations

$$VX = \omega_0(x\sin\alpha - y\cos\alpha),$$

$$VY = \omega_0(x\cos\alpha + y\sin\alpha),$$

so that the Fourier transform representation of ψ may be written in the form

$$\psi = \iint F(K_1, K_2) \exp\{i(K_1 X + K_2 Y)\}\, dK_1 dK_2, \tag{1}$$

where $K_2 = (K_2 \cos\alpha + K_1 \sin\alpha)\,(K_1^2 + K_2^2 + \beta^2)^{-\frac{1}{2}}$ and F is related to f in an obvious way.

The implications of the radiation condition are most readily seen by following Lighthill's (1967) rule, which states that the condition is satisfied by only those waves which lie in those parts of the plane which are covered by the normals to

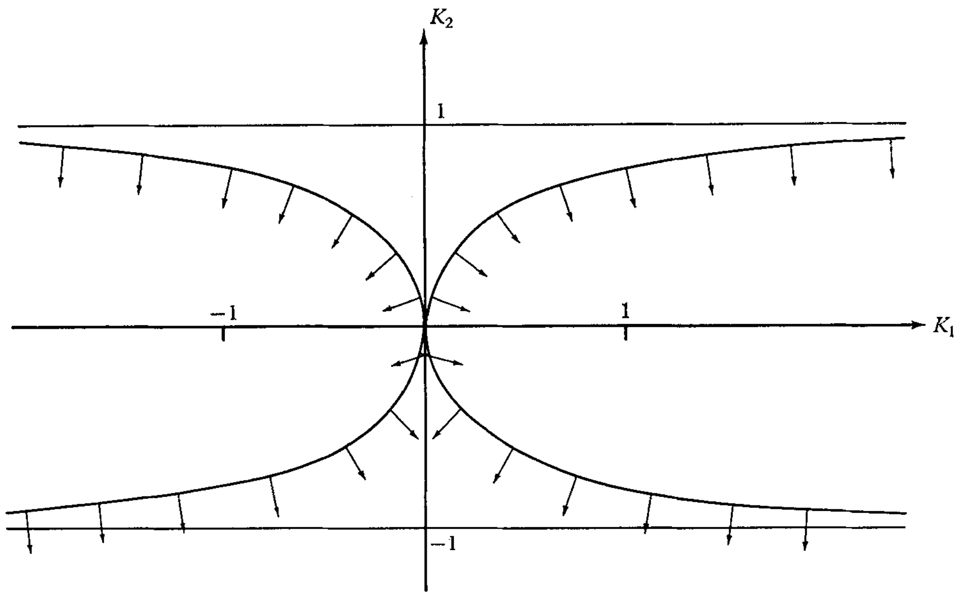

FIGURE 1. The curve $\omega = 0$ for $\alpha = 90°$, $\beta = 0$.

the curve $\omega = 0$, drawn in the direction of increasing ω. The curves $\omega = 0$ however are a two-parameter set. We shall find that the main features of the wave patterns for arbitrary β can be deduced from the curves $\omega = 0$ with $\beta = 0$. The equation for the curve $\omega = 0$ can be written in the form

$$(K^2 + \beta^2)^{\frac{1}{2}} = |\sin(\theta + \alpha)/\sin\theta|$$

where we put $K_1 = K\cos\theta$ and $K_2 = K\sin\theta$. Curves $\omega = 0$, $\beta = 0$ for $\alpha = 90°$, $30°$ and $1°$ are shown in figures 1–3; the direction of the arrows indicates the direction of normals pointing towards increasing ω. When $\alpha = 90°$, figure 1, so that the body is moving vertically upwards, no arrows point into the first or second quadrants, so that no waves are propagated ahead of the body. When $\alpha = 30°$, figure 2, both branches of the curve have a point of inflexion at the origin and a common tangent at an angle $-\alpha$ to the K_1 direction. Since K_2 is conjugate to the direction of motion of the body which is inclined at the angle α to the horizontal, no arrows lie in the wedge between the line of motion of the body and the positive x-axis; therefore no waves will be found in this wedge. There are, however, waves ahead of the body in the fourth quadrant of the (x, y)-plane. In figure 3, $\alpha = 1°$, there are no waves in the wedge of 1° to the right of the positive K_2-axis. In figures 1–3 it will be noted that arrows never point into the second quadrant so that waves will not be found in the conjugate directions in the (x, y)-plane. If

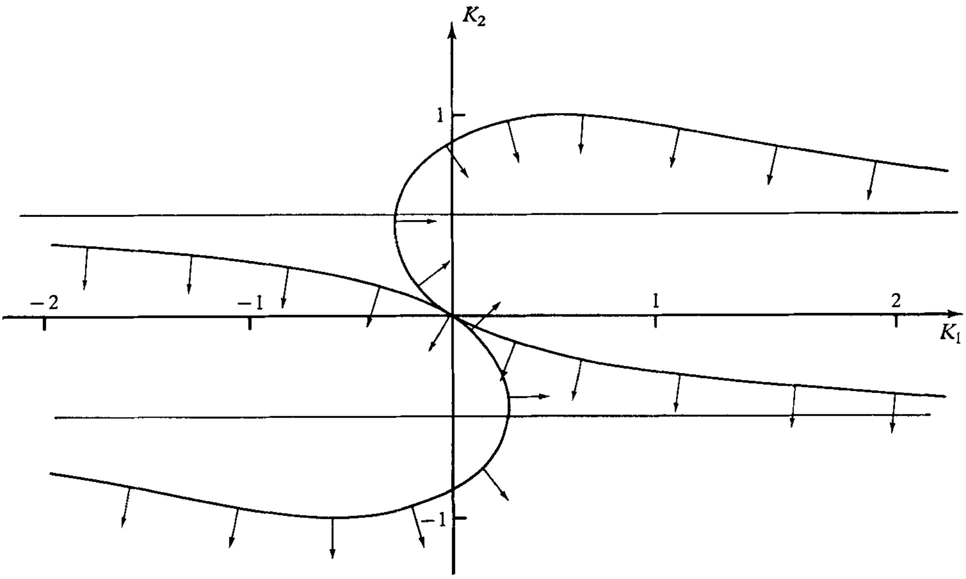

FIGURE 2. The curve $\omega = 0$ for $\alpha = 30°$, $\beta = 0$.

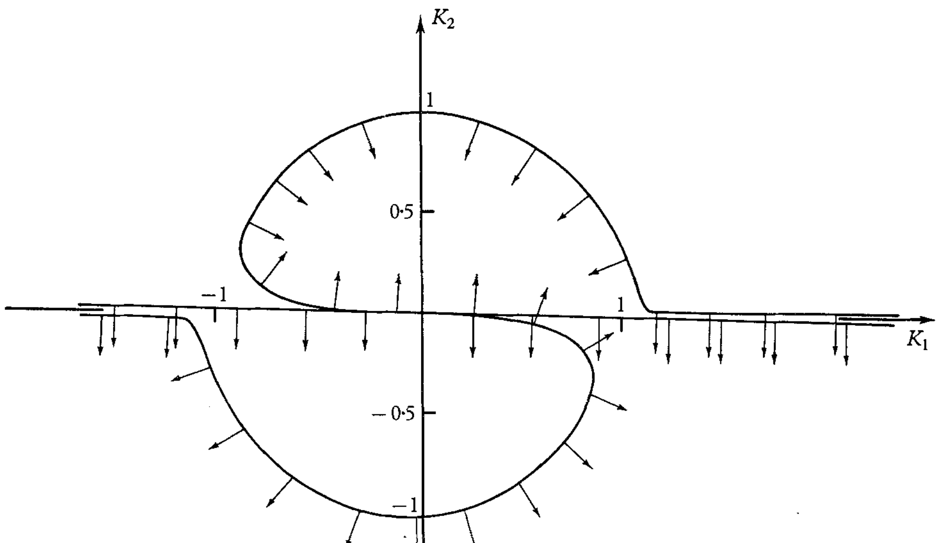

FIGURE 3. The curve $\omega = 0$ for $\alpha = 1°$, $\beta = 0$.

$\alpha = 0$, then the curve $\omega = 0$, $\beta = 0$ becomes the unit circle in the K-plane together with the K_1-axis and there are arrows in the direction of positive K_2 corresponding to $K_2 = 0$, $0 \leqslant |K_1| \leqslant 1$; this is shown in figure 4.

Let us now return to the general case $\beta \neq 0$; we see that the curve $\omega = 0$, $\beta \neq 0$ may be derived from $\omega = 0$, $\beta = 0$ by reducing the square of the radial distance to any point on the curve by an amount β^2. Therefore those points on the diagram $\omega = 0$, $\beta = 0$ which lie inside a circle centred at the origin with radius β will have no corresponding points on the diagram $\omega = 0$, $\beta \neq 0$. It will have been noticed that in the case $\beta = 0$ the waves which propagate ahead of the disturbance all correspond to wave-numbers somewhat less than unity; the extreme case is

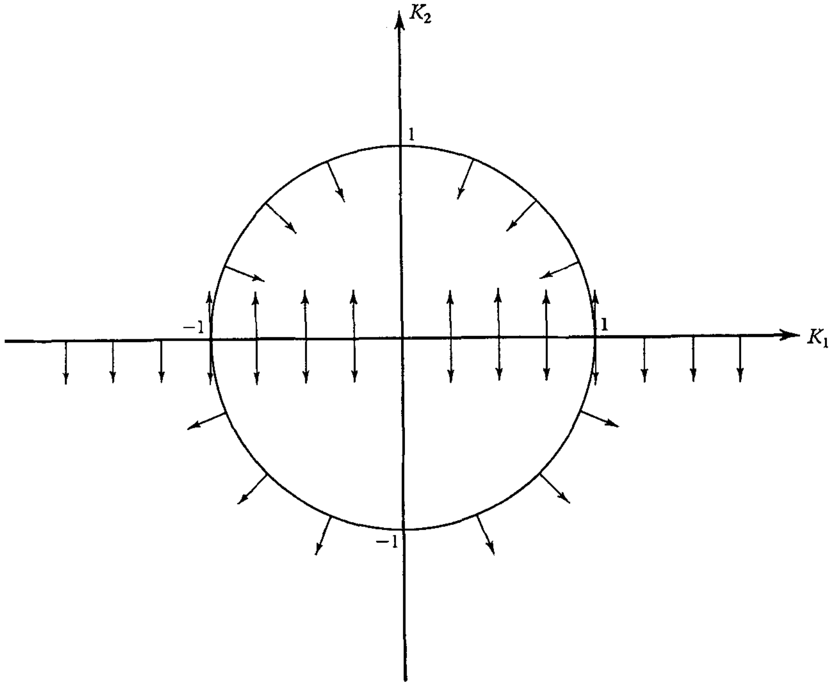

FIGURE 4. The curve $\omega = 0$ for $\alpha = 0$, $\beta = 0$.

that of $\alpha = 0$, $\beta = 0$. Thus we can say that the effect of β's being non-zero is to remove at least a portion of the curve $\omega = 0$ along which arrows point into the first K quadrant and may well ensure that no arrows point into that quadrant. Note that this result has not appealed in any way to the finite extent of the disturbance.

If we apply arguments of stationary phase to obtain an asymptotic value of the integral

$$\psi = \iint F(K_1, K_2) \exp\{i(K_1 X + K_2 Y)\} dK_1 dK_2$$

with $(K^2+\beta^2)^{\frac{1}{2}} = |\sin(\theta+\alpha)/\sin\theta|$, $\theta = \tan^{-1} K_2/K_1$, then the points of stationary phase are given by

$$XY^{-1} dK_1/dK_2 + 1 = 0, \quad X/Y = O(1),$$

which simplifies to give

$$\left\{\cot\theta - \frac{\dfrac{\sin^2(\theta+\alpha)}{\sin^2\theta} - \beta^2}{\sin(\theta+\alpha)\cos(\theta+\alpha) - \beta^2 \sin\theta\cos\theta}\right\} X = -Y.$$

Points of constant phase are

$$K[X\cos\theta + Y\sin\theta] = \Phi, \quad \text{say,}$$

which yields

$$\frac{X}{\Phi} = \frac{\sin(\theta+\alpha)\cos(\theta+\alpha) - \beta^2 \sin\theta\cos\theta}{\sin\theta\left[\dfrac{\sin^2(\theta+\alpha)}{\sin^2\theta} - \beta^2\right]^{\frac{3}{2}}},$$

$$\frac{Y}{\Phi} = \frac{1}{\sin\theta\left[\dfrac{\sin^2(\theta+\alpha)}{\sin^2\theta} - \beta^2\right]^{\frac{1}{2}}} - \frac{\sin(\theta+\alpha)\cos(\theta+\alpha)\cot\theta - \beta^2\cos^2\theta}{\sin\theta\left[\dfrac{\sin^2(\theta+\alpha)}{\sin^2\theta} - \beta^2\right]^{\frac{3}{2}}}.$$

Points of double stationary phase correspond to the lines $X = 0$, $Y = 0$ and $\theta = \theta_0$. If $\beta \equiv 0$, θ_0 is the root of

$$\tan(\theta+\alpha) = \cot\theta \pm (\cot^2\theta - 1)^{\frac{1}{2}},$$

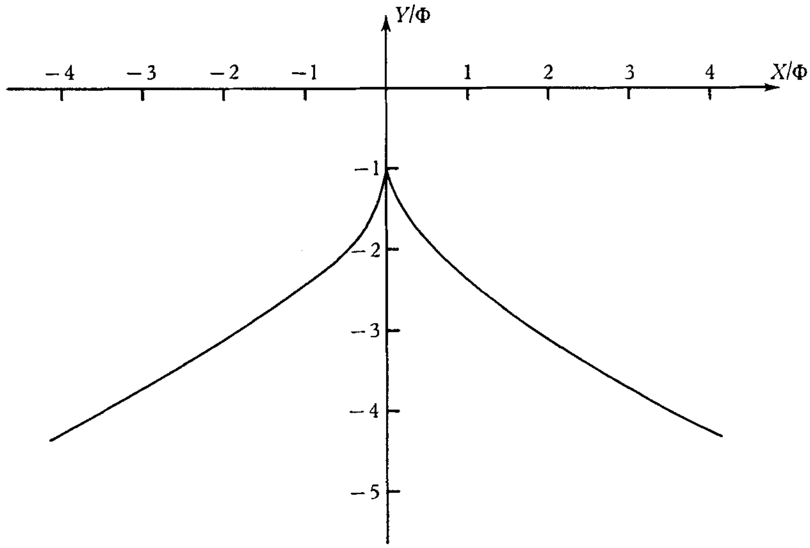

FIGURE 5. A line of constant phase for $\alpha = 90°$, $\beta = 0$.

which always has a solution and only one solution. The pattern of lines of constant phase for $\alpha = 90°$, $\beta = 0$ is shown in figure 5. The corresponding axisymmetric problem for $\alpha = 90°$ is that of an ascending sphere, a situation in which the wave pattern differs in no essential respect from the two-dimensional case, has been studied and was described by Mowbray & Rarity (1967*b*): the agreement between experiment and theory was good. The pattern has a series of cusps on the line of motion behind the body, no crest passing through the disturbance. Figure 6 shows the pattern for $\alpha = 30°$. We see that there are no waves in the wedge between the Y-axis and the x-axis; the Y-axis corresponds to the line of motion of the body. The crest is tangential to the positive and negative x-axis at infinity. There is a caustic along the trailing portion of the line of motion of the body and a line of cusps on the line $Y/X = \tan 79°$, that is the line $y/x = \tan 19°$. Successive crests will interfere in the wedge bounded by the line of cusps and the caustic in the third quadrant of the (X, Y)-plane. Figure 7 shows a constant phase line for $\alpha = 1°$; the body is moving at $1°$ to the horizontal, 'gravity' acts from left to right in the figure. As $\alpha \to 0$, the portions AB, AC move off towards larger and larger negative X and Y in such a way that the point A moves towards the line

$Y/X = 1$, leaving a single line along the whole Y-axis; the accumulation of lines of constant phase on the Y-axis constitutes the upstream and downstream wake, and is a singular case.

We note that in all wave patterns other than $\alpha = 90°$ the crests and troughs tend asymptotically to the x-axis so that there is always an accumulation of waves on the x-axis, far upstream and downstream.

If $\beta \neq 0$, the form of lines of constant phase is much the same; there is still a

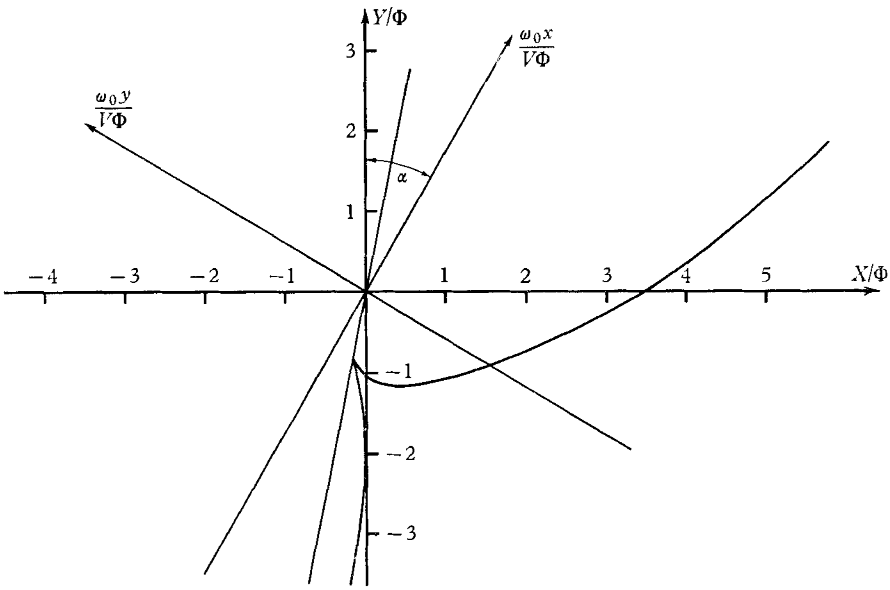

FIGURE 6. A line of constant phase for $\alpha = 30°$, $\beta = 0$.

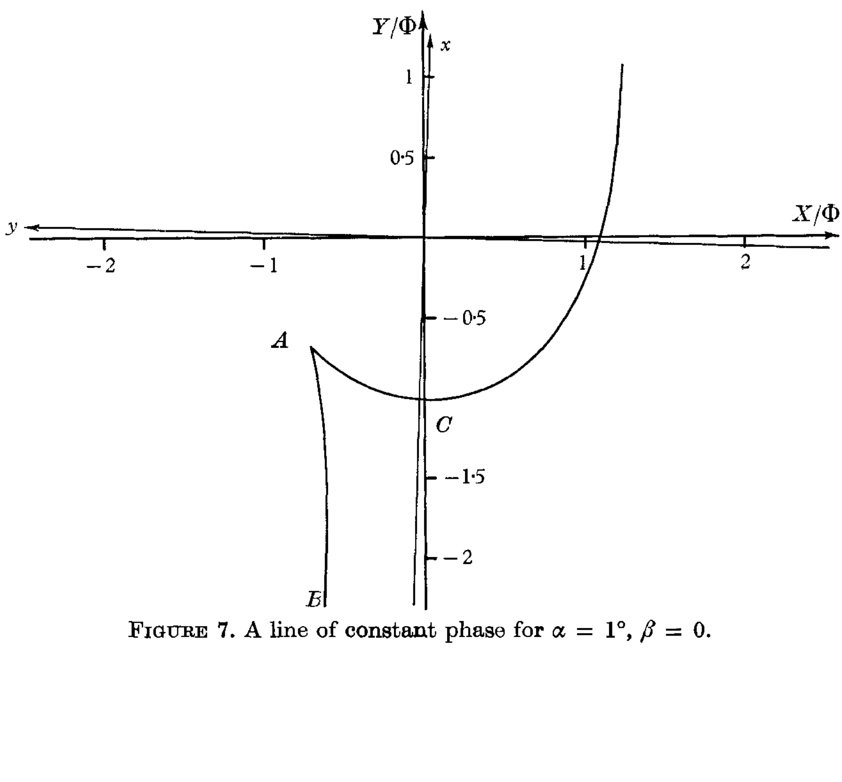

FIGURE 7. A line of constant phase for $\alpha = 1°$, $\beta = 0$.

caustic corresponding to $\sin\theta = 0$, at $X/\Phi = 0$, $Y/\Phi = -1/\sin\alpha$. There are no waves in the wedge of angle $\tan^{-1}[\sin\alpha/(\cos\alpha-\beta)]$ in the positive x-direction lying below the line of motion of the disturbance. The wedge now covers a portion of the fourth quadrant in the (x, y)-plane. The boundary of the wedge which lies in the fourth quadrant is an asymptote for all lines of constant phase; in addition, its reflexion in the origin is an asymptote. There is still an accumulation of crests and troughs along the asymptotes.

We may study the singular nature of these accumulations by considering the expression (1) in more detail. If in (1) the variables are changed to K and θ and $X = R\cos\phi$, $Y = R\sin\phi$, then

$$\psi = \int_0^\infty \int_0^{2\pi} \frac{F^*(K,\theta)\exp\{iRK\cos(\theta-\phi)\}K^{-1}dK\,d\theta}{\sin^2\theta(K^2+\beta^2)-\sin^2(\theta+\alpha)},$$

where $F^*(K,\theta)$ represents the effect of the finite extent of the disturbance. To accomplish the integration with respect to K explicitly, take

$$F^*(K,\theta) = K\exp\{-AK\}$$

corresponding to a disturbance proportional to $d/dR(R^2+A^2)^{-1}$. Then

$$\psi = \sim \frac{\pi i}{2}\int_0^{2\pi} \frac{\exp\{iR\cos(\theta-\phi)\,K(\theta)\}\exp\{-AK(\theta)\}}{\sin^2\theta K(\theta)}\,d\theta,$$

where
$$K(\theta) = \left\{\frac{\sin^2(\theta+\alpha)}{\sin^2\theta}-\beta^2\right\}^{\frac{1}{2}},$$

and we have neglected a term of order $\exp\{-KR|\cos(\theta-\phi)|\}$; $K(\theta)$ is required to be real. For sufficiently large R, the main contribution to the integral will arise from the zeros of the denominator and the points at which $\cos(\theta-\phi)K(\theta)$ is stationary. The contribution from the zeros of $\sin\theta$ is zero by virtue of the fact that $K(\theta)\to\infty$ and A is non-zero. The corresponding point on the wave is the caustic along the trailing portion of the line of motion of the body. This singularity then is smoothed out by the finite extent of the disturbance. On the other hand, the contribution from the zeros of $K(\theta)$ is finite and independent of R for sufficiently large R. The corresponding points are those portions of the wave which are asymptotic to the line

$$Y/X = \tan^{-1}(\sin\alpha/(\cos\alpha-\beta)).$$

The stationary values of $\cos(\theta-\phi)K(\theta)$ establish a relation between θ and ϕ which is precisely the relation between the direction θ in figures 1–4 and the direction ϕ of the positive normals to the curves. The contribution is $O(R^{-\frac{1}{2}})$ except at the double points, where it is $O(R^{-\frac{1}{3}})$ and is essentially unaffected by the finite extent of the disturbance.

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